

ANALYSIS OF THE $\bar{\partial}$ -NEUMANN PROBLEM ALONG A STRAIGHT EDGE

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ABSTRACT. We show there exists an L^p solution, for $p \in (2, \infty)$, to the $\bar{\partial}$ -Neumann problem on an edge domain in \mathbb{C}^2 for $(0, 1)$ -forms, and we explicitly compute the singularities, which are of complex logarithmic and arctangent type, along the edge, of the solution.

1. INTRODUCTION

The aim of this paper is to provide insight into the singular behavior of solutions of the $\bar{\partial}$ -Neumann problem on domains which are not smooth. We consider the edge domain, $\Omega \subset \mathbb{C}^2$, defined by

$$\{(z_1, z_2) \in \mathbb{C}^2 : \Im z_1 > \alpha \Im z_2 > 0\}$$

for some $0 \leq \alpha < \infty$, and solutions to the $\bar{\partial}$ -Neumann problem on Ω for $(0, 1)$ -forms. A solution to the $\bar{\partial}$ -Neumann problem is an inverse to the complex Laplacian, $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, on Ω .

The results obtained here are a generalization of results in [2], which deals with the case of $\alpha = 0$, in which the singularities of the solution are explicitly computed. Other properties of the Neumann operator on non-smooth domains are described in Ehsani [1], Engliš [3], Henkin and Iordan [5], Henkin, Iordan, and Kohn [6], Michel and Shaw [7, 8], and Straube [9].

The domain of the edge considered here is an important model domain in the study of the $\bar{\partial}$ -Neumann problem on non-smooth domains because, as in [2] and [1], we can compute explicitly the singularities in the solution, however, on the edge, the problem has the added complexity that the two components u_1 and u_2 of the $(0, 1)$ -form solution $u = u_1 d\bar{z}_1 + u_2 d\bar{z}_2$ are coupled. We resolve this difficulty by examining the boundary conditions in detail along the edge. The domain is also important in that it depends on a parameter, α . Thus this domain should serve better as a prototype for a wider class of non-smooth domains.

2. FINDING A SOLUTION

We consider the $\bar{\partial}$ -Neumann problem on an edge, Ω in \mathbb{C}^2 described by

$$\{(z_1, z_2) \in \mathbb{C}^2 : \Im z_1 > \alpha \Im z_2 > 0\}$$

for some $0 \leq \alpha < \infty$. The case of $\alpha = 0$, in which Ω is the cross product of two half-planes, was studied in detail in [2]. For our data $(0, 1)$ -form, f , we make the assumption $f \in \mathcal{S}_{(0,1)}(\bar{\Omega})$, the space of $(0, 1)$ -forms whose coefficients are Schwartz functions. We use the notation $z_j = x_j + iy_j$, for $j = 1, 2$. On the interior of Ω the $\bar{\partial}$ -Neumann problem becomes

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$$\Delta u_j = -2f_j \quad j = 1, 2,$$

and the boundary conditions are

$$u_2 = 0, \quad \frac{\partial u_1}{\partial \bar{z}_2} = 0 \quad \text{on } y_2 = 0,$$

and

$$u_1 - \alpha u_2 = 0, \quad \frac{\partial u_1}{\partial \bar{z}_2} - \frac{\partial u_2}{\partial \bar{z}_1} = 0 \quad \text{on } y_1 = \alpha y_2.$$

We make the change of coordinates

$$\begin{aligned} Y_2 &= y_2 \\ Y_1 &= y_1 - \alpha y_2, \end{aligned}$$

and we define the functions

$$\begin{aligned} u_\alpha &= u_1 - \alpha u_2 \\ f_\alpha &= f_1 - \alpha f_2. \end{aligned}$$

In these new coordinates the interior equations become

$$(2.1) \quad \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + (1 + \alpha^2) \frac{\partial^2}{\partial Y_1^2} - 2\alpha \frac{\partial^2}{\partial Y_1 \partial Y_2} + \frac{\partial^2}{\partial Y_2^2} \right) u_j = -2f_j$$

for $j = \alpha, 2$, on the product of two half-planes, $\mathbb{H} \times \mathbb{H} = \{(x_1, x_2, Y_1, Y_2) : Y_1, Y_2 > 0\}$, and the boundary conditions become

$$(2.2) \quad u_\alpha = 0 \quad \text{on } Y_1 = 0,$$

$$(2.3) \quad u_2 = 0 \quad \text{on } Y_2 = 0,$$

$$(2.4) \quad -i\alpha \frac{\partial u_\alpha}{\partial Y_1} = \frac{\partial u_2}{\partial x_1} - \alpha \frac{\partial u_2}{\partial x_2} + i \left((1 + \alpha^2) \frac{\partial u_2}{\partial Y_1} - \alpha \frac{\partial u_2}{\partial Y_2} \right) \quad \text{on } Y_1 = 0,$$

$$(2.5) \quad \frac{\partial u_\alpha}{\partial x_2} + i \left(-\alpha \frac{\partial u_\alpha}{\partial Y_1} + \frac{\partial u_\alpha}{\partial Y_2} \right) = -i\alpha \frac{\partial u_2}{\partial Y_2} \quad \text{on } Y_2 = 0.$$

We apply the Fourier transform to (2.1) on the domain $\mathbb{H} \times \mathbb{H}$. We transform the equation for u_α .

$$(2.6) \quad \begin{aligned} & -(\lambda_1^2 + \lambda_2^2 + (1 + \alpha^2)\eta_1^2 - 2\alpha\eta_1\eta_2 + \eta_2^2)\hat{u}_\alpha \\ & - (1 + \alpha^2) \frac{\partial \tilde{u}_\alpha}{\partial Y_1} \Big|_{Y_1=0} + i(2\alpha\eta_1 - \eta_2)\tilde{u}_\alpha \Big|_{Y_2=0} - \frac{\partial \tilde{u}_\alpha}{\partial Y_2} \Big|_{Y_2=0} \\ & = -2\hat{f}_\alpha, \end{aligned}$$

where λ_j is the transform variable corresponding to x_j and η_j is the transform variable corresponding to Y_j for $j = 1, 2$, and \tilde{u}_j denotes the partial transform in all variables except Y_1 and $\tilde{\tilde{u}}_j$ denotes the partial transform of u_j in all variables except Y_2 .

We use the superscript, oj , to denote an odd reflection with respect to Y_j . Reflecting (2.6) to be odd in η_1 , we have

$$(2.7) \quad \begin{aligned} & -(\lambda_1^2 + \lambda_2^2 + (1 + \alpha^2)\eta_1^2 - 2\alpha|\eta_1|\eta_2 + \eta_2^2)\hat{u}_\alpha^{o1} \\ & + i(2\alpha|\eta_1| - \eta_2)\tilde{\tilde{u}}_\alpha^{o1} \Big|_{Y_2=0} - \frac{\partial \tilde{\tilde{u}}_\alpha^{o1}}{\partial Y_2} \Big|_{Y_2=0} = -2\hat{f}_\alpha^{o1}. \end{aligned}$$

We use (2.5) to eliminate $\frac{\partial \tilde{u}_\alpha^{o1}}{\partial Y_2} \Big|_{Y_2=0}$ from equation 2.7:

$$(2.8) \quad -(\lambda_1^2 + \lambda_2^2 + (1 + \alpha^2)\eta_1^2 - 2\alpha|\eta_1|\eta_2 + \eta_2^2)\hat{u}_\alpha^{o1} \\ + i(\alpha|\eta_1| - \eta_2 - i\lambda_2)\tilde{u}_\alpha^{o1} \Big|_{Y_2=0} + \alpha \frac{\partial \tilde{u}_2^{o1}}{\partial Y_2} \Big|_{Y_2=0} = -2\hat{f}_\alpha^{o1}.$$

We let $\zeta_1 = \sqrt{\eta_1^2 + \lambda^2}$ and set $\eta_2 = \alpha|\eta_1| - i\zeta_1$ in (2.8) in order to eliminate $\tilde{u}_\alpha^{o1} \Big|_{Y_2=0}$.

Finally we solve for \hat{u}_α^{o1} in terms of \hat{f}_α and $\frac{\partial \tilde{u}_2^{o1}}{\partial Y_2} \Big|_{Y_2=0}$;

$$(2.9) \quad \hat{u}_\alpha^{o1} = -i\alpha \frac{1}{\eta_2 - \alpha|\eta_1| - i\zeta_1} \frac{1}{\lambda_2 - \zeta_1} \frac{\partial \tilde{u}_2^{o1}}{\partial Y_2}(\lambda_1, \lambda_2, \eta_1, 0) \\ + \frac{2}{\lambda_1^2 + \lambda_2^2 + (1 + \alpha^2)\eta_1^2 - 2\alpha|\eta_1|\eta_2 + \eta_2^2} \times \\ \left(\hat{f}_\alpha^{o1} - \frac{\lambda_2 + i(\alpha|\eta_1| - \eta_2)}{\lambda_2 - \zeta_1} \hat{f}_\alpha^{o1}(\lambda_1, \lambda_2, \eta_1, \alpha|\eta_1| - i\zeta_1) \right).$$

Following an analogous procedure, we write

$$\hat{u}_2^{o2} = i\alpha \frac{1}{(1 + \alpha^2)\eta_1 - \alpha|\eta_2| + i\zeta_2} \frac{1}{\lambda_1 - \alpha\lambda_2 - \zeta_2} \frac{\partial \tilde{u}_\alpha^{o2}}{\partial Y_1}(\lambda_1, \lambda_2, 0, \eta_2) \\ + \frac{2}{\lambda_1^2 + \lambda_2^2 + (1 + \alpha^2)\eta_1^2 - 2\alpha\eta_1|\eta_2| + \eta_2^2} \times \\ \left(\hat{f}_2^{o2} - \frac{\lambda_1 - \alpha\lambda_2 + i\alpha|\eta_2| - i(1 + \alpha^2)\eta_1}{\lambda_1 - \alpha\lambda_2 - \zeta_2} \hat{f}_2^{o2}(\lambda_1, \lambda_2, \frac{\alpha|\eta_2| - i\zeta_2}{1 + \alpha^2}, \eta_2) \right).$$

An examination of the consistency of the boundary conditions (2.4) and (2.5) along the edge reveals

$$(2.10) \quad \frac{\partial u_2}{\partial Y_2} \Big|_{Y_1=Y_2=0} = \frac{\partial u_\alpha}{\partial Y_1} \Big|_{Y_1=Y_2=0}.$$

Furthermore, relations (2.1), (2.2), and (2.3) in (2.4) when $Y_2 = 0$ allow us to determine $\frac{\partial^k}{\partial Y_1^k} \frac{\partial u_2}{\partial Y_2} \Big|_{Y_1=Y_2=0}$ ($k \geq 1$) in terms of $\frac{\partial^{k-1}}{\partial Y_1^{k-1}} \frac{\partial u_2}{\partial Y_2} \Big|_{Y_1=Y_2=0}$. Similarly, relations (2.1), (2.2), and (2.3) in (2.5) when $Y_1 = 0$ give $\frac{\partial^k}{\partial Y_2^k} \frac{\partial u_\alpha}{\partial Y_1} \Big|_{Y_1=Y_2=0}$ ($k \geq 1$) in terms of $\frac{\partial^{k-1}}{\partial Y_2^{k-1}} \frac{\partial u_\alpha}{\partial Y_2} \Big|_{Y_1=Y_2=0}$. Thus, in the case $u \in C_{(0,1)}^1(\bar{\Omega})$, $\frac{\partial^k}{\partial Y_1^k} \frac{\partial u_2}{\partial Y_2} \Big|_{Y_1=Y_2=0}$ and $\frac{\partial^k}{\partial Y_2^k} \frac{\partial u_\alpha}{\partial Y_1} \Big|_{Y_1=Y_2=0}$ are finite $\forall k \geq 0$.

Remark 2.1. By considering the decay of (2.9) with respect to the Fourier variables, as they go to ∞ , from the condition $u \in C_{(0,1)}^1(\bar{\Omega})$ we can conclude that

$$\frac{\partial u_2}{\partial Y_2} \Big|_{Y_1=Y_2=0} = \frac{\partial u_\alpha}{\partial Y_1} \Big|_{Y_1=Y_2=0} = 0$$

The last two terms of (2.9) represent terms in $C^1(\bar{\Omega})$, hence the decay of the first must be sufficient enough to eliminate lower order terms (see [2] for details of this argument).

In the case $u \in C^1_{(0,1)}(\overline{\Omega})$, the finite Taylor coefficients, combined with the fact that

$$\frac{\partial u_2}{\partial Y_2}(x_1, x_2, Y_1, 0) \in C^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$$

(see the arguments in Corollary 2.5 below), shows us

$$\frac{\partial u_2}{\partial Y_2}(x_1, x_2, Y_1, 0) \in C^\infty(\mathbb{R}^2 \times \overline{\mathbb{R}_+}).$$

From the symmetry of the domain in the x -variables and the fact that $f \in \mathcal{S}_{(0,1)}(\overline{\Omega})$, we assume $\frac{\partial u_2}{\partial Y_2}\big|_{Y_2=0}$ is Schwartz with respect to the x variables, and, so that the partial Fourier transform is determined (up to a $C^\infty(\overline{\Omega})$ term) by the Taylor coefficients at $Y_1 = 0$, we also assume $\frac{\partial u_2}{\partial Y_2}\big|_{Y_2=0}$ is Schwartz with respect to Y_1 , and thus that $\frac{\partial u_2}{\partial Y_2}\big|_{Y_2=0} \in \mathcal{S}(\mathbb{R}^2 \times \overline{\mathbb{R}_+})$.

We are therefore led to choose a $b_2(x_1, x_2, Y_1) \in \mathcal{S}(\mathbb{R}^2 \times \overline{\mathbb{R}_+})$ which agrees to infinite order with $\frac{\partial u_2}{\partial Y_2}(x_1, x_2, Y_1, 0)$ at $Y_1 = 0$, possible by Borel's theorem. We also can choose in an analogous manner a $b_\alpha(x_1, x_2, Y_2) \in \mathcal{S}(\mathbb{R}^2 \times \overline{\mathbb{R}_+})$ which agrees to infinite order with $\frac{\partial u_\alpha}{\partial Y_1}(x_1, x_2, 0, Y_2)$ at $Y_2 = 0$. The singular terms in the solution we obtain are independent of the choice of b_2 and b_α , as the next lemma will show.

Definition 2.2. We say $\hat{h}_1(\xi_1, \xi_2, \eta_1, \eta_2)$ is equivalent to $\hat{h}_2(\xi_1, \xi_2, \eta_1, \eta_2)$, or $\hat{h}_1 \sim \hat{h}_2$, if

$$h_1 - h_2\big|_\Omega \in C^\infty(\overline{\mathbb{H} \times \mathbb{H}}).$$

Lemma 2.3. Assume $u \in C^1_{(0,1)}(\overline{\Omega})$ and further that $\frac{\partial u_2}{\partial Y_2}\big|_{Y_2=0} \in \mathcal{S}(\mathbb{R}^2 \times \overline{\mathbb{R}_+})$. Let $b_2 \in \mathcal{S}(\mathbb{R}^2 \times \overline{\mathbb{R}_+})$ be chosen as described above, then

$$(2.11) \quad \frac{1}{\eta_2 - \alpha|\eta_1| - i\zeta_1} \frac{1}{\lambda_2 - \zeta_1} \frac{\partial \tilde{u}_2^{o1}}{\partial Y_2}(\lambda_1, \lambda_2, \eta_1, 0) \sim \frac{1}{\eta_2 - \alpha|\eta_1| - i\zeta_1} \frac{1}{\lambda_2 - \zeta_1} \tilde{b}_2^{o1}(\lambda_1, \lambda_2, \eta_1).$$

Also the relation, (2.11), is independent of the choice of b_2 .

In the proof of Lemma 2.11 we use the notation \lesssim to mean $\leq c$ for $c > 0$.

Proof. We first show

$$\frac{1}{\eta_2 - \alpha|\eta_1| - i\zeta_1} \frac{1}{\lambda_2 - \zeta_1} \tilde{b}_2^{o1}(\lambda_1, \lambda_2, \eta_1) \in L^p(\mathbb{R}^4)$$

for $p \in (1, 2)$. First integrating over η_2 , we consider

$$\int_{\mathbb{R}^4} \left| \frac{1}{\eta_2 - \alpha|\eta_1| - i\zeta_1} \frac{1}{\lambda_2 - \zeta_1} \tilde{b}_2^{o1}(\lambda_1, \lambda_2, \eta_1) \right|^p d\lambda d\eta \lesssim \int_{\mathbb{R}^3} \left| \frac{\tilde{b}_2^{o1}(\lambda_1, \lambda_2, \eta_1)}{\lambda_2 - \zeta_1} \right|^p \frac{1}{\zeta_1^{p-1}} d\lambda d\eta_1,$$

where $d\lambda = d\lambda_1 d\lambda_2$ and $d\eta = d\eta_1 d\eta_2$. Changing $(\eta_1, \lambda_1, \lambda_2)$ to polar coordinates, (r, ϕ, θ) , we then estimate

$$(2.12) \quad \int_0^{2\pi} \int_0^\pi \int_0^\infty \left| \frac{\tilde{b}_2^{o1}(\lambda_1, \lambda_2, \eta_1)}{1 - \cos \phi} \right|^p \frac{\sin \phi}{r^{2p-3}} dr d\phi d\theta.$$

It is elementary to show, from the fact that $b_2 \in \mathcal{S}(\mathbb{R}^2 \times \overline{\mathbb{R}}_+)$ and, from Remark 2.1, which gives $b_2(Y_1 = 0) = 0$, that

$$(2.13) \quad \left| \frac{\partial}{\partial \eta_1} \tilde{b}_2^{o1}(\lambda_1, \lambda_2, \eta_1) \right| \lesssim \frac{1}{1 + r^3}.$$

Therefore, with the Fundamental Theorem of Calculus, and $\tilde{b}_2^{o1}(\lambda_1, \lambda_2, 0) = 0$, (2.13) gives us the estimate

$$\tilde{b}_2^{o1}(\lambda_1, \lambda_2, \eta_1) \lesssim \frac{1}{1 + r^3} |\eta_1|,$$

which, when used in (2.12), shows convergence of the integral.

Now, if $v(x_1, x_2, Y_1) \in \mathcal{S}(\mathbb{R}^2 \times \overline{\mathbb{R}}_+)$, and v vanishes to infinite order at $Y_1 = 0$ then after using a partial Fourier inverse with respect to η_2 of

$$(2.14) \quad \frac{1}{\eta_2 - \alpha|\eta_1| - i\zeta_1} \frac{1}{\lambda_2 - \zeta_1} \tilde{v}^{o1}(\lambda_1, \lambda_2, \eta_1),$$

we can use the decay of $v^{o1}(\lambda_1, \lambda_2, \eta_1)$, faster than any power of $1/\eta_1$, to show (2.14) is actually the transform of a function which, when restricted to $\mathbb{H} \times \mathbb{H}$, is in $C^\infty(\overline{\mathbb{H} \times \mathbb{H}})$. We denote by FT_2 the partial Fourier transform with respect to Y_2 , and Φ to be the Fourier inverse of (2.14).

$$(2.15) \quad \begin{aligned} |\zeta_1|^j \widehat{\frac{\partial^k}{\partial Y_2^k} \Phi} &= |\zeta_1|^j FT_2 \left(\frac{\partial^k}{\partial Y_2^k} \tilde{\Phi} \right) \\ &= |\zeta_1|^j (i\alpha|\eta_1| - \zeta_1)^k \widehat{\Phi} \\ &\lesssim \frac{1}{\eta_2 - \alpha|\eta_1| - i\zeta_1} \frac{1}{\lambda_2 - \zeta_1} |\zeta_1|^{j+k} \tilde{v}^{o1}(\lambda_1, \lambda_2, \eta_1). \end{aligned}$$

Taking into account the decay of $\tilde{v}^{o1}(\lambda_1, \lambda_2, \eta_1)$, we can show (2.15) is in $L^p(\mathbb{R}^4)$ following the same proof for $j = k = 0$ above.

We then prove the lemma by setting

$$v = \frac{\partial u_2}{\partial Y_2} \Big|_{Y_2=0} - b_2$$

above. □

As a corollary we have the

Proposition 2.4. *Let u_α and u_2 be defined on $\mathbb{H} \times \mathbb{H}$ in terms of their Fourier transforms as*

$$(2.16) \quad \hat{u}_\alpha^{o1} = -i\alpha \frac{1}{\eta_2 - \alpha|\eta_1| - i\zeta_1} \frac{1}{\lambda_2 - \zeta_1} \tilde{b}_2^{o1}(\lambda_1, \lambda_2, \eta_1) \\ + \frac{2}{\lambda_1^2 + \lambda_2^2 + (1 + \alpha^2)\eta_1^2 - 2\alpha|\eta_1|\eta_2 + \eta_2^2} \times \\ \left(\hat{f}_\alpha^{o1} - \frac{\lambda_2 + i(\alpha|\eta_1| - \eta_2)}{\lambda_2 - \zeta_1} \hat{f}_\alpha^{o1}(\lambda_1, \lambda_2, \eta_1, \alpha|\eta_1| - i\zeta_1) \right)$$

and

$$(2.17) \quad \hat{u}_2^{o2} = i\alpha \frac{1}{(1 + \alpha^2)\eta_1 - \alpha|\eta_2| + i\zeta_2} \frac{1}{\lambda_1 - \alpha\lambda_2 - \zeta_2} \tilde{b}_\alpha^{o2}(\lambda_1, \lambda_2, \eta_2) \\ + \frac{2}{\lambda_1^2 + \lambda_2^2 + (1 + \alpha^2)\eta_1^2 - 2\alpha\eta_1|\eta_2| + \eta_2^2} \times \\ \left(\hat{f}_2^{o2} - \frac{\lambda_1 - \alpha\lambda_2 + i\alpha|\eta_2| - i(1 + \alpha^2)\eta_1}{\lambda_1 - \alpha\lambda_2 - \zeta_2} \hat{f}_2^{o2}(\lambda_1, \lambda_2, \frac{\alpha|\eta_2| - i\zeta_2}{1 + \alpha^2}, \eta_2) \right).$$

Then u_α and u_2 are in $L^p(\mathbb{H} \times \mathbb{H})$ for $p \in (2, \infty)$.

Proof. The first terms of the Fourier transforms, (2.16) and (2.17), are in $L^p(\mathbb{R}^4)$ from the proof of Lemma 2.3. The proof that the last two terms in (2.16) and (2.17) are in $L^p(\mathbb{R}^4)$ is the same as in the case of $\alpha = 0$ (see [2]). Then the Proposition follows by the Hausdorff-Young theorem relating L^p estimates of functions in terms of L^p estimates of their transforms. \square

Corollary 2.5. *Let u_α and u_2 be defined as in Proposition 2.4. Then u_α and u_2 are in $C^\infty(\bar{V})$ for all neighborhoods $V \subset \mathbb{H} \times \mathbb{H}$ such that V does not intersect $\{Y_1 = 0\} \cap \{Y_2 = 0\}$.*

Proof. We present the proof for u_α . Interior regularity follows from the strong ellipticity of the Laplacian.

Also, general regularity at the boundary arguments for the Dirichlet problem can be applied to the case in which V is a neighborhood such that $V \cap \partial(\mathbb{H} \times \mathbb{H}) = V \cap \{Y_1 = 0\} \neq \emptyset$ (see [4]).

If V is a neighborhood which intersects $Y_2 = 0$, then the tangential derivatives commute with the $\bar{\partial}$ -Neumann problem in V , and thus as above, we can show $D_\tau^k u_\alpha \in L^p(V)$ when $p > 2$, for all tangential derivatives D_τ^k of all orders k . Furthermore, since u_2 and $\frac{\partial u_1}{\partial \bar{z}_2} - \frac{\partial u_2}{\partial \bar{z}_1}$, in Ω , satisfy Dirichlet conditions along $y_2 = 0$, after a transformation, they belong to $C^\infty(\bar{V})$, and hence we can also derive estimates involving normal derivatives, and we conclude $D^k u_\alpha \in L^p(V)$ when $p > 2$ for all derivatives D^k of all orders k . Hence, a Sobolev embedding theorem applies to prove the corollary. \square

With u_α and u_2 defined on $\mathbb{H} \times \mathbb{H}$ as in Proposition 2.4, and with $u_1 = u_\alpha + \alpha u_2$, we also denote by $u_1(x_1, x_2, y_1, y_2)$ and $u_2(x_1, x_2, y_1, y_2)$ the corresponding functions, defined on Ω , under the transformation $y_1 = Y_1 + \alpha Y_2$ and $y_2 = Y_2$.

Theorem 2.6. *With $u_1(x_1, x_2, y_1, y_2)$ and $u_2(x_1, x_2, y_1, y_2)$ defined as above, the $(0, 1)$ -form, $u = u_1 d\bar{z}_1 + u_2 d\bar{z}_2$, is in $C_{(0,1)}^1(\bar{\Omega})$ and $L_{(0,1)}^p(\Omega)$, for $p \in (2, \infty)$, and solves the $\bar{\partial}$ -Neumann problem on the edge, Ω , with data, $f \in \mathcal{S}_{(0,1)}(\bar{\Omega})$. This*

solution is unique in the sense that any other $C^1_{(0,1)}(\bar{\Omega})$ solution in $L^p_{(0,1)}(\Omega)$ and whose boundary terms, $\frac{\partial u_2}{\partial y_2}(y_2 = 0)$ and $\left(\frac{\partial u_1}{\partial y_1} - \alpha \frac{\partial u_2}{\partial y_1}\right)(y_1 = \alpha y_2)$, are in $\mathcal{S}(\mathbb{R}^2 \times \overline{\mathbb{R}_+})$, differs by a function in $C^\infty(\bar{\Omega})$.

Proof. Remark 2.1 shows $u \in C^1_{(0,1)}(\bar{\Omega})$, and Proposition 2.4 shows $u \in L^p_{(0,1)}(\Omega)$. That u solves the $\bar{\partial}$ -Neumann problem follows by our construction at the beginning of Section 2.

The uniqueness part of the Proposition also follows from Remark 2.1 which shows any solution in $C^1_{(0,1)}(\bar{\Omega})$ and $L^p_{(0,1)}(\Omega)$ is determined by choices of b_2 and b_α , which are unique modulo functions which vanish to infinite order at $Y_1 = 0$ and $Y_2 = 0$, respectively, and thus, from Lemma 2.3 the solutions are unique modulo functions in $C^\infty(\bar{\Omega})$. \square

3. SINGULARITIES

We shall examine the type of singularities which are present in the solution described in Theorem 2.6. We shall proceed as in [2], expanding \hat{u}_α^{o1} and \hat{u}_2^{o2} as asymptotic series for large $|\eta_1|$ and $|\eta_2|$, in which higher order terms correspond to a class of functions on $\mathbb{H} \times \mathbb{H}$ of greater differentiability, continuous up to the boundary. We work with u_α , the analysis being similar for u_2 . In what follows, for $j = 1, 2$, let $\chi_{\eta_j}(\eta_j)$ be an even, smooth function of η_j , with the property $\chi_{\eta_j} = 1$ for $|\eta_j| < a$ and $\chi_{\eta_j} = 0$ for $|\eta_j| > b$ for some $b > a > 0$. Also, define $\chi_\eta(\eta_1, \eta_2)$ to be a smooth function of η_1 and η_2 , even in both variables, with the property $\chi_\eta = 1$ for $\eta_1^2 + \eta_2^2 < a$ and $\chi_\eta = 0$ for $\eta_1^2 + \eta_2^2 > b$ for some $b > a > 0$.

Lemma 3.1. *For $j = 1, 2$ let $\chi'_{\eta_j} = 1 - \chi_{\eta_j}$, and let $\chi'_\eta = 1 - \chi_\eta$. With the equivalence relation defined in Definition 2.2,*

$$\hat{u}_\alpha^{o1} \sim \chi'_{\eta_1} \chi'_{\eta_2} \hat{u}_\alpha^{o1}.$$

Sketch of proof. The equivalence between \hat{u}_α^{o1} and $\chi'_\eta \hat{u}_\alpha^{o1}$ is obvious, and that between $\chi'_\eta \hat{u}_\alpha^{o1}$ and $\chi'_{\eta_1} \chi'_{\eta_2} \hat{u}_\alpha^{o1}$ may be shown by evaluating decay properties of $(\chi'_\eta - \chi'_{\eta_1} \chi'_{\eta_2}) \hat{u}_\alpha^{o1}$ in Fourier transform space.

$$(3.1) \quad \chi'_\eta - \chi'_{\eta_1} \chi'_{\eta_2} = (\chi_{\eta_1} - \chi_\eta) + \chi_{\eta_2} (1 - \chi_{\eta_1}).$$

When the first term on the right hand side of (3.1) is multiplied by each term of \hat{u}_α^{o1} , as expressed by (2.16), after taking a partial Fourier inverse with respect to η_2 , we can use the relation between taking derivatives with respect to Y_2 and multiplying by ζ_1 as in (2.15) to show differentiability in all variables given the decay with respect to the Fourier variables λ_1 , λ_2 and η_1 .

When the second term on the right hand side of (3.1) is multiplied by each term of \hat{u}_α^{o1} , as expressed by (2.16), we can again use the relation between taking derivatives with respect to Y_2 and multiplying by ζ_1 , this time using decay with respect to η_2 to derive decay with respect to η_1 to finish the proof of the lemma. \square

To obtain our asymptotic expansion of $\chi'_{\eta_1} \chi'_{\eta_2} \hat{u}_\alpha^{o1}$, we expand ζ_1 for large $|\eta_1|$,

$$\frac{1}{\lambda_1^2 + \lambda_2^2 + (1 + \alpha^2)\eta_1^2 - 2\alpha|\eta_1|\eta_2 + \eta_2^2} = \frac{1}{\lambda_1^2 + \lambda_2^2 + (\eta_2 - \alpha|\eta_1|)^2 + \eta_1^2}$$

as a geometric series in $(\eta_2 - \alpha|\eta_1|)^2 + \eta_1^2$, and we integrate by parts all Fourier integrals involving f_α or b_2 , leaving as remainders those terms which decay faster than either of

$$\frac{1}{|\eta_1|^{2n+3}} \frac{1}{\lambda_1^2 + \lambda_2^2 + (\eta_2 - \alpha|\eta_1|)^2 + \eta_1^2}$$

or

$$\frac{1}{|\eta_2|^{2n+3}} \frac{1}{\lambda_1^2 + \lambda_2^2 + (\eta_2 - \alpha|\eta_1|)^2 + \eta_1^2}$$

for large $|\eta_1|$ or $|\eta_2|$. Again relating Y_2 derivatives of the partial Fourier inverse of $\frac{1}{\lambda_1^2 + \lambda_2^2 + (\eta_2 - \alpha|\eta_1|)^2 + \eta_1^2}$ with multiplication by ζ_1 , we can show decay in $|\eta_1|$ gives differentiability with respect to Y_2 and vice-versa, and thus all remainder terms are Fourier transforms of functions in $C^n(\mathbb{H} \times \mathbb{H})$.

Our asymptotic expansion, for large $|\eta_1|$, $|\eta_2|$, is thus a sum of terms of the form

$$(3.2) \quad \chi'_{\eta_1} \chi'_{\eta_2} c_{jklm}(\lambda_1, \lambda_2) \frac{1}{\eta_2^j} \frac{1}{|\eta_1|^k} \frac{1}{\eta_1^{2l-1}} \frac{1}{((\eta_2 - \alpha|\eta_1|)^2 + \eta_1^2)^m}$$

for $j \geq -1$, $k, m = 0, 1$, and $l, n \geq 1$, where $c_{jklm}(\lambda_1, \lambda_2)$ are in $\mathcal{S}(\mathbb{R}^2)$.

We start with the terms

$$\frac{\chi'_{\eta_1} \chi'_{\eta_2}}{((\eta_2 - \alpha|\eta_1|)^2 + \eta_1^2)^{j+1}}.$$

For $\eta_1 \neq 0$ and $Y_2 > 0$

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{((\eta_2 - \alpha|\eta_1|)^2 + \eta_1^2)^{j+1}} e^{i\eta_2 Y_2} d\eta_2 = \\ \frac{2\pi i}{j!} \sum_{k=0}^j \binom{j}{k} (-1)^{j-k} \frac{(2j-k)!}{j!} \frac{(iY_2)^k}{(2i|\eta_1|)^{2j-k+1}} e^{|\eta_1|(-1+i\alpha)Y_2}, \end{aligned}$$

which is a linear combination of terms of the form

$$\int_0^{Y_2} \cdots \int_0^{t_2} \int_{-\infty}^{\infty} \frac{1}{(\eta_2^2 + \eta_1^2)^l} e^{i\eta_2(1-i\alpha)Y_2} d\eta_2 dt_1 \cdots dt_{j+1-l}.$$

Such terms (excluding the constants of integration, whose inverses are singular along all of $Y_2 > 0$) were studied in Lemma 3.7 of [2]. And from the same Lemma 3.7, which gives $\frac{\chi'_{\eta_1} \chi'_{\eta_2}}{\eta_1^2 + \eta_2^2}$ locally near $Y_1 = Y_2 = 0$, we immediately have

Lemma 3.2. *The inverse Fourier transform of*

$$\frac{\chi'_{\eta_1} \chi'_{\eta_2}}{((\eta_2 - \alpha|\eta_1|)^2 + \eta_1^2)^{j+1}},$$

near $Y_1 = Y_2 = 0$, has the form

$$(3.3) \quad p(Y_1, Y_2) \log(Y_1^2 + (1-i\alpha)^2 Y_2^2),$$

where p is a homogeneous polynomial of degree $2j$ in Y_1 and Y_2 , modulo functions which are in $C^\infty(\mathbb{H} \times \mathbb{H})$ or are singular along all of $Y_2 > 0$.

With a slight abuse of notation we shall use the equivalence relation in Definition 2.2 to apply to functions defined on $\mathbb{H} \times \mathbb{H}$.

We now define functions Φ_l on $Y_2 \geq 0$ which have the form of (3.3) such that

$$\chi'_{\eta_1} \chi'_{\eta_2} \widehat{\chi \Phi_l} \sim \frac{\chi'_{\eta_1} \chi'_{\eta_2}}{((\eta_2 - \alpha|\eta_1|)^2 + \eta_1^2)^l}.$$

Then with Φ_l defined for $l \geq 1$, we define $(\Phi_l)_0 = \Phi_l$ for $Y_2 \geq 0$, and, for $j \geq 1$, $(\Phi_l)_j$ to be the unique solution of the form

$$p_1 \log(Y_1^2 + (1 - i\alpha)^2 Y_2^2) + p_2 + p_3 \arctan \left(\frac{Y_1}{(1 - i\alpha)Y_2} \right)$$

on the half-plane $\{(Y_1, Y_2) : Y_2 \geq 0\}$, where p_1 , p_2 , and p_3 are polynomials in Y_1 and Y_2 such that $p_2(0, Y_2) = 0$, to the equation

$$\frac{\partial(\Phi_l)_j}{\partial Y_1} = (\Phi_l)_{j-1}.$$

Also, define for $k \geq 1$, on $Y_2 \geq 0$ and restricting to $Y_1 \geq 0$,

$$(\Phi_l)_{jk} = \int_0^{Y_2} \cdots \int_0^{t_2} \int_0^{t_1} (\Phi_l)_j(Y_1, t) dt dt_1 \cdots dt_{k-1}.$$

Then integration by parts in the Fourier transform integral shows

$$\chi'_{\eta_1} \chi'_{\eta_2} \widehat{\chi(\Phi_l)_{jk}^{o1}} \sim \frac{\chi'_{\eta_1} \chi'_{\eta_2}}{|\eta_1|^m \eta_1^{2n+1} \eta_2^k} \frac{1}{((\eta_2 - \alpha|\eta_1|)^2 + \eta_1^2)^l},$$

where $2n + 1 + m = j$.

We are now ready to prove the

Theorem 3.3. *Let $f \in S_{(0,1)}(\bar{\Omega})$, and $u = u_1 d\bar{z}_1 + u_2 d\bar{z}_2$ be the $(0,1)$ -form which solves the $\bar{\partial}$ -Neumann problem on Ω with data f . Then, in Ω , near $y_1 = y_2 = 0$, u_j can be written as*

$$(3.4) \quad u_j = \alpha_{j1} \log((y_1 - \alpha y_2)^2 + (1 - i\alpha)^2 y_2^2) + \alpha_{j2} \log((y_1 - \alpha y_2)^2 (1 - i\alpha)^2 + y_2^2) \\ + \beta_{j1} \arctan \left(\frac{y_1 - \alpha y_2}{(1 - i\alpha)y_2} \right) + \beta_{j2} \arctan \left(\frac{(1 - i\alpha)(y_1 - \alpha y_2)}{y_2} \right) + \gamma_j,$$

where α_{jk} , β_{jk} and γ_j are smooth for $j, k = 1, 2$.

Proof. We may use the functions $(\Phi_l)_{jk}$ constructed above, which have the form

$$(\Phi_l)_{jk} = p_1 \log(Y_1^2 + (1 - i\alpha)^2 Y_2^2) + p_2 + p_3 \arctan \left(\frac{Y_1}{(1 - i\alpha)Y_2} \right) + p_4 \log|Y_1|,$$

where the p_m are homogeneous polynomials of degree $(2l - 2) + j + k$ in Y_1 and Y_2 for $m = 1, 2, 3, 4$, to see the structure of the terms of the form

$$\frac{\chi'_{\eta_1} \chi'_{\eta_2}}{|\eta_1|^m \eta_1^{2n+1} \eta_2^k} \frac{1}{((\eta_2 - \alpha|\eta_1|)^2 + \eta_1^2)^l}$$

arising in the asymptotic expansion for u_α . For the other terms, of the form,

$$\chi'_{\eta_1} \chi'_{\eta_2} \frac{\eta_2}{|\eta_1|^m \eta_1^{2n+1}} \frac{1}{((\eta_2 - \alpha|\eta_1|)^2 + \eta_1^2)^l}$$

we use the property

$$\frac{\partial \Phi_l}{\partial Y_2} = \frac{(1 - i\alpha)^2}{2(j-1)} y_2 \Phi_{l-1}.$$

Since Corollary 2.5 allows us to conclude any singular terms along all of $Y_1 = 0$ or $Y_2 = 0$ must vanish, we can take a finite number of terms of the form (3.2) in the asymptotic expansion and pair each with an appropriate function constructed with the $(\Phi_l)_{jk}$, ignoring singular terms such as $\log |y_1|$ to show $\forall n \in \mathbb{N}$, \exists polynomials, A_n , B_n , and C_n , of degree n in Y_1 and Y_2 , and whose coefficients are Schwartz functions of λ_1 and λ_2 , and D_n , the partial transform in the x variables of a function which belongs to $C^n(\overline{\mathbb{H} \times \mathbb{H}})$, such that near $Y_1, Y_2 = 0$

$$F.T._x(u_\alpha^{o1})(\lambda_1, \lambda_2, Y_1, Y_2) = A_n \log(Y_1^2 + (1 - i\alpha)^2 Y_2^2) + B_n + C_n \arctan\left(\frac{Y_1}{(1 - i\alpha)Y_2}\right) + D_n,$$

where $F.T._x$ stands for the partial Fourier transform in the x variables.

Lastly, using Borel's theorem, inverting with respect to λ_1 and λ_2 , and transforming back to the variables, y_1 and y_2 , we can show u_α is of the form (3.4). Then combining with an analogous argument applied to u_2 , we conclude the theorem. \square

We end with the note that Theorem 3.3 is non-trivial; there exists an $f \in \mathcal{S}_{(0,1)}(\overline{\Omega})$, for instance an $f \in \mathcal{S}_{(0,1)}(\overline{\Omega})$ which is equivalently equal to 1 in a neighborhood of the edge, such that one of the α_{ij} or β_{ij} is not equivalently 0.

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